

THE ALEPH-ZERO OR ZERO DICHOTOMY

(New and extended version with new arguments)

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ABSTRACT. This paper proves the existence of a dichotomy which being formally derived from the topological successiveness of ω^* -order leads to the same absurdity of Zeno's Dichotomy II. It also derives a contradictory result from the first Zeno's Dichotomy.

1. INTRODUCTION: ZENO'S PARADOXES AND MODERN SCIENCE

Zeno's Paradoxes have interested philosophers of all times¹ although until the middle of the XIX century they were frequently considered as mere sophisms. From that time, and particularly through the XX century, they became the unending source of new philosophical, mathematical and physical discussion. Authors as Hegel James, Russell, Whitehead or Bergson² focused their attention on the challenging world of Zeno's paradoxes. At the beginning of the second half of the XX century the pioneering works of Black, Wisdom, Thomson, and Benacerraf³ introduced a new way of discussing the possibilities of performing an actual infinity of actions in a finite time (a performance involved in most of Zeno's paradoxes). I refer to Supertask Theory. In fact, infinity machines, or supermachines, are our modern Achilles substitutes. A supermachine is a theoretical device supposedly capable of performing countably many actions in a finite interval of time. The possibilities of performing an uncountable infinity of actions were ruled out by P. Clark and S. Read, for which they made use of Cantor's argument on the impossibility of dividing a real interval into uncountably many adjacent parts.⁴ Although supertasks have also

¹See [14], [15], [83], [73], [47], [84], [25] or [56] for historical background.

²[43], [48], [72], [86], [87], [9], [10].

³[12], [88], [79], [80], [8].

⁴[23].

been examined from the perspective of nonstandard⁵ analysis, as far as I know the possibilities to perform an hypertask along an hyperreal interval of time have not been discussed, although finite hyperreal intervals can be divided into hypercountably many successive infinitesimal intervals, the so called hyperfinite partitions.⁶ Supertask theory has finally turned its attention, particularly from the last decade of the XX century, towards the discussion of the physical plausibility of supertasks as well as on the implications of supertasks in the physical world including relativistic and quantum mechanics perspectives.⁷

In the second half of the XX century, several solutions to some of Zeno's paradoxes have been proposed. Most of those solutions were found in the context of new branches of mathematics as Cantor's transfinite arithmetic, topology, measure theory and more recently internal set theory⁸ (a branch of nonstandard analysis). It is also worth noting the solutions proposed by P. Lynds⁹ within a classical and quantum mechanics framework. Some of these solutions, however, have been contested. And in most of cases the proposed solutions do not explain where Zeno's arguments fail. Moreover, some of the proposed solutions gave rise to a new collection of problems so exciting as Zeno's paradoxes.¹⁰

The four most famous Zeno's paradoxes are usually regarded as arguments against motion¹¹ be it performed in a continuous or in a discontinuous world. *Achilles and the Tortoise* and the *Dichotomy* in the continuous case, *the Stadium* and *the Arrow* in the discontinuous one. The paradoxes of the second case (together with the paradox of Plurality) are more difficult to solve, if a solution exists after all, particularly in a quantum spacetime framework. Most of the proposed solutions to Zeno's paradoxes are, in effect, solutions to the paradoxes of the first group or to the second one in a dense and continuous spacetime framework. This situation is very significant taking into account the increasing number of contemporary physical theories suggesting the quantum

⁵[58], [57], [1], [54].

⁶[77], [34], [50], [44], etc.

⁷[69], [63], [67], [73], [39], [41], [40], [64], [65], [31], [66], [61], [2], [3], [68], [85], [45], [29], [30], [28], [74].

⁸[37], [38], [91], [39], [41], [40], [58], [57].

⁹[52], [53].

¹⁰[62], [1], [67], [73], [47] [74].

¹¹[4], [38], [42], [24], [73] etc.

nature of spacetime, as for instance superstring theory, loop quantum gravity, quantum computation theory, or black hole thermodynamics.¹² Will physics (the science of change) finally meet the problem of change (whose insolubility motivated Zeno's argument) at this quantum level¹³? Is the problem of Change really inconsistent as some authors¹⁴ have defended? These are in fact two intriguing and still unsolved questions related to Zeno's arguments.¹⁵

2. ZENO'S PARADOXES AND ω -ORDER

Not less intriguing, though for different reasons, is the fact that one immediately perceives when examining the contemporary discussions on Zeno's paradoxes. Surprisingly, the Axiom of Infinity is never involved in such discussions. Zeno's arguments have never been used to question the Axiom of Infinity, as if the existence of actual infinite totalities were beyond any doubt.¹⁶ Grünbaum, for instance, proposed that if it were the case that from modern kinematics together with the denseness postulate a false zenonian conclusion could be formally derived, then we would have to replace current kinematics by other mechanical theory.¹⁷ Anything but questioning the hypothesis of the actual infinity from which the involved topological denseness derives. And this in spite of the lack of evidence of that hypothesis, which is even rejected by some schools of contemporary mathematics as constructivism (among whose precursors we find scholars as Newton, Fermat or Euler¹⁸) and by some XX century thinkers of the intellectual stature of Poincaré or Wittgenstein.¹⁹

In the first half of the XIX century Bernard Bolzano, and in the second one Richard Dedekind, tried unsuccessfully²⁰ to prove the existence of infinite totalities.²¹ For his part, G. Cantor, the founder of transfinite mathematics, simply took it for granted the existence of such totalities. Thus, in §6 of his famous *Beiträge* (pp. 103-104 of the English translation) we can read:

¹²[35], [36], [81], [32], [75], [5], [76], [78], [6], [51], [6], [78].

¹³[7].

¹⁴[59], [60].

¹⁵[62].

¹⁶[49].

¹⁷[38, page 39].

¹⁸[55].

¹⁹[70], [89].

²⁰Their respective proofs were compatible with the potential infinity..

²¹[13], [27].

The first example of a transfinite set is given by the totality of finite cardinals.

although, as could be expected, he gave no proof on the the existence of that totality. In accordance with his profound theological platonism, Cantor was firmly convinced of the actual existence of complete infinite totalities.²² He never explicitly declared the hypothetical nature of his infinitist assertions (at least not in his most relevant works on the transfinite.²³ He even tried to prove the existence of actual infinities (quoted in [71], p. 3, from [19], p. 404):

... in truth the potential infinite has only a borrowed reality, insofar as potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on.

Evidently this is not a formal proof but a personal belief. Cantor's infinite totality is isomorph to the set \mathbb{N} of natural numbers and then his implicit assumption on the existence of that complete totality is equivalent to our modern Axiom of Infinity.

The (assumed) infinite totality of finite cardinals led Cantor to the essential notion of ω -order (Beiträge, p. 115 [20]):

By ω we understand the type of a well ordered aggregate:

$$(e_1, e_2, \dots, e_\nu, \dots)$$

in which:

$$e_\nu \prec e_{\nu+1}$$

and where ν represents all finite cardinal numbers in turn.

Cantor then defined the notion of *fundamental series of ordinals* of which he proved the existence of a limit (Beiträge, Theorem §14 I). This limit plays a capital role in the proofs of the following 10 theorems in Beiträge §15 the last of which is the fundamental theorem K-15 stating that every ordinal of the second class (transfinite) is either the result of increasing by one the next smaller ordinal (ordinals of the first kind), or the limit of a fundamental increasing sequence of ordinals (ordinals of the second kind). Cantor construction of transfinite ordinals, from ω to the ϵ -numbers of the second number class, strongly

²²[26].

²³[17], [18], [16], [21], [22].

depends on Theorem K-15. The imposing cantorian edifice was really founded on that theorem. And that theorem, in turn, depends on the hypothetical existence of a complete infinite totality: that of the finite cardinal numbers (Axiom of Infinity in modern terms), which is anything but selfevident.

In modern terms we say a sequence is ω -ordered if it has a first element and each element has an immediate successor. Similarly, a sequence is ω^* -ordered if it has a last element and each element has an immediate predecessor. Evidently, both type of ordering are intimately related to Zeno's Dichotomies, although surprisingly, the analysis of Zeno's arguments as formal consequences of ω -order remains still undone. For some unknown reasons, it seems we are not interested in analyzing the consistency of the hypothetical existence of actual complete infinite totalities. And this in spite of the enormous problems the actual infinity poses to experimental sciences as physics (recall for example the problems of renormalization in elementary particle physics²⁴). The discussion that follows is just oriented in that direction. Its main objective is to analyze Zeno's Dichotomies I and II from the perspective of ω -order and ω^* -order respectively.

3. THE ALEPH-ZERO OR ZERO DICHOTOMY

In what follows, and for the sake of clarity, I will consider a canonical version of the famous Achilles' race whose logical impossibility Zeno claimed. In fact, Achilles will be considered as a single mass point moving rightwards along the X axis, from point -1 to point 1, at a finite velocity v . In the place of the uncountable and densely ordered sequence of points within the real interval $[-1, 1]$ we will only consider the ω^* -ordered sequence of points:

$$\dots, \frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2}, 1 \quad (1)$$

all of which Achilles must successively traverse in order to reach point 1 from the starting point -1. In fact, this denumerable sequence of points (Z^* -points according to classical Vlastos' terminology²⁵) is not densely by successively ordered, which means that between any two successive Z^* -points no other Z^* -point exists. In consequence, and at a finite velocity, Z^* -points can only be traversed in a successive way: one after the other. Assume now Achilles is just on the point 0 at the precise instant t_0 . According to classic mechanics he will

²⁴[33], [46], [35], [90], [36].

²⁵[82].

reach point 1 just at²⁶ $t_1 = t_0 + 1/v$. But before reaching his goal, he has to successively traverse the controversial Z^* -points. We will focus our attention just on the way Achilles performs such a traversal. For this, let $f_{z^*}(t)$ be the number of Z^* -points Achilles has traversed at the precise instant t , being t any instant within the closed real interval $[t_0, t_1]$. It is quite clear that $f_{z^*}(t_0) = 0$ because at t_0 Achilles is just on point 0. For any other instant t in $[t_0, t_1]$ Achilles has already passed over countably many Z^* -points, for if there were an instant t in $[t_0, t_1]$ at which Achilles were passed only over a finite number $n > 0$ of Z^* -points, these n Z^* -points would have to be the impossible firsts n points of an ω^* -ordered sequence of points. So we can write:

$$f_{z^*}(t) = \begin{cases} 0 & \text{if } t = t_0 \\ \aleph_0 & \text{if } t_0 < t \leq t_1 \end{cases} \quad (2)$$

Notice $f_{z^*}(t)$ is well defined for each t in $[t_0, t_1]$. Consequently, f_{z^*} maps the real interval $[t_0, t_1]$ into the set of two elements $\{0, \aleph_0\}$. In this way f_{z^*} defines a clair dichotomy, the *aleph-zero or zero dichotomy*,²⁷ regarding the numbers of Z^* -points Achilles has traversed when moving rightward from -1 to 1 along the X axis. Accordingly, with respect to the number of the traversed Z^* -points, Achilles can only exhibit two states:

- (1) State A_0 : Achilles has traversed zero Z^* -points.
- (2) State A_{\aleph_0} : Achilles has traversed aleph-zero Z^* -points.

Thus, Achilles directly becomes from having traversed no Z^* -point (state A_0) to having traversed \aleph_0 of them (state A_{\aleph_0}). Finite intermediate states, as A_n at which he would have traversed only a finite number n of Z^* -points, simply do not exist. The set of states Achilles exhibits with respect to the number of traversed Z -points is well defined and has only two elements, namely A_0 and A_{\aleph_0} . Let us now examine the transition from A_0 to A_{\aleph_0} under the inevitable restriction of the above *aleph-zero or zero dichotomy*. The topological successiveness of Z^* -points makes it impossible to traverse them other than successively. And taking into account that between any two successive Z^* -points a finite distance greater than 0 exists, to traverse \aleph_0 of those Z^* -points

²⁶Assuming, for instance, that v is given in kilometers per second and that the distance from 0 to 1 is just one kilometer..

²⁷Although the usual way of reading \aleph_0 is aleph-null -it can also be read as aleph-zero- the original English translation by P. E. B. Jourdain of Cantor's *Beiträge* was "aleph-zero". Section 6 is entitled "The Smallest Transfinite Cardinal Number Aleph-Zero. The current English edition of Cantor's *Beiträge* is from 1955..

-whatever they be- means to traverse a finite distance greater than 0. This traversal, at the finite Achilles' velocity v , can only be accomplished by lasting a certain amount of time necessarily greater than 0. Achilles, therefore, has to expend a certain amount of time $\tau > 0$ in becoming A_{\aleph_0} from A_0 . The ω^* -ordering imposes this time τ has to be indeterminable, otherwise we would know the precise instant at which Achilles becomes A_{\aleph_0} and, consequently, we would also know the precise Z^* -point point on which he reaches that state, and this is impossible because in that case we would have a natural number n such that $n + 1 = \aleph_0$. The indeterminacy of τ means both the existence of more than one alternative and the impossibility to determine the actual alternative. Now then, indeterminable as it may be, τ must be greater than 0, and this inevitable requirement imposed by the fact that Achilles must traverse a distance greater than 0 at its finite velocity v is incompatible with the aleph-zero or zero dichotomy, as we will immediately see.

In fact, let τ be any real number greater than zero and assume the transition from A_0 to A_{\aleph_0} lasts a time τ . Consider the real interval $(0, \tau)$ and any instant $t \in (0, \tau)$. At t Achilles state cannot be neither A_0 nor A_{\aleph_0} . It cannot be A_0 because if that were the case the process of becoming A_{\aleph_0} would not have begun, and then the process of becoming A_{\aleph_0} would last a time equal or less than $\tau - t$ in the place of the assumed τ . It cannot be A_{\aleph_0} because in that case the process of becoming A_{\aleph_0} would have already finished and then it would have lasted a time equal or less than t in the place of the assumed τ . Now then, Achilles' state has to be either A_0 or A_{\aleph_0} because it is well defined along the real interval $[0, 1]$ of which $(0, \tau)$ is a proper subinterval. Consequently, and being τ any real number, it is impossible for Achilles to become A_{\aleph_0} from A_0 by lasting a time greater than zero. Notice this is not a question of indeterminacy but of impossibility: no real number greater than zero exists for the duration of Achilles' transition from A_0 to A_{\aleph_0} . He, therefore, has to become A_{\aleph_0} from A_0 instantaneously. But this is impossible at his finite velocity v . He must, therefore, remain A_0 . Or in other words, he cannot begin to move. Evidently, this conclusion is the same absurdity of Zeno's Dichotomy II, although in our case it has been directly derived from the topological successiveness of ω^* -order, which in turn derives from assuming the existence of complete denumerable totalities (Axiom of infinity), as Cantor proved.²⁸ To be complete (as the actual infinity requires) and uncompletable (because no last -first- element completes

²⁸[20].

them) could be a contradictory attribute rather than a permissible eccentricity of both ω -ordered and ω^* -ordered sequences.

The above conclusion is confirmed by the following argument.²⁹ Let us replace each Z^* -point with a mass Z^* -sensor capable of emitting a visible laser beam when it is activated by any mass passing over it. Assume the system of Z^* -sensors is regulated in such a way that each sensor emits its corresponding laser beam if, and only if, it is activated and no other laser beam is being emitted by other Z^* -sensor of the system. So only one laser beam can be being emitted by the system of Z^* -sensors: the one corresponding to the first activated Z^* -sensor, whatsoever it be. Assume Achilles performs his canonical race from point -1 to point 1. Will any laser beam being emitted at t_1 ? Evidently not, because it would have to be the impossible first Z^* -sensor of an ω^* -ordered sequence of Z^* -sensors. But on the other hand, why not? Is there any reason to explain the inevitable malfunctioning of the Z^* -sensors system other than the inconsistency of assuming that it is possible to begin a sequence of discrete and successive actions without a first action to begin? What is impossible is not motion but the actual infiniteness of ω^* -order.

Let us now examine Zeno's Dichotomy I under the same canonical conditions of the above Dichotomy II. Consider again the real interval $[-1, 1]$ in the X axis. Let now $\langle z_i \rangle_{i \in \mathbb{N}}$ be the ω -ordered sequence of Z -points:

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots, \frac{2^i - 1}{2^i}, \dots \quad (3)$$

Achilles has to traverse in his race from point -1 to point 1. Assume also we remove from $(0, 1)$ all points except just Z -points (we would have a sort of *Zeno's powder*). In the place of a continuous race from point -1 to point 1, assume that Achilles is on point 0 just at instant t_0 and then he begins to jump to z_1 , to z_2 , to z_3 , \dots , and to any point x in $[1, 2]$ if there were no other Z -points to jump, in such a way that he is on each z_i just at t_i as a consequence of a Z -jump j_i , being t_i the i -th term of the ω -ordered sequence of instants $\langle t_i \rangle_{i \in \mathbb{N}}$ whose limit is t_b .

²⁹A variant of Benardete's paradox [11].

The one to one correspondence $f(t_i) = z_i$ proves³⁰ that at t_b Achilles has completed the ω -ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ on the ω -ordered sequence of Z-points $\langle z_i \rangle_{i \in \mathbb{N}}$. Thus, at t_b Achilles has to be on a point $x \geq 1$ of $[1, 2]$. Otherwise, if he were on a Z-point z_i , only a finite number i of jumps would have been performed. We now have an uncomfortable asymmetry between the ω -ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ and the $(\omega + 1)$ -ordered sequence of points: the ω -ordered sequence of Z-points plus the last point x Achilles ends up his ω -ordered sequence of Z-jumps, i.e. the $(\omega + 1)$ -ordered sequence $\langle \langle z_i \rangle_{i \in \mathbb{N}}, x \rangle$.

By definition Achilles is on each z_i at t_i as a consequence of the i -th Z-jump j_i . The one to one correspondence $f(j_i) = z_i$ proves that:

- (1) Each z-jump j_i ends on a z-point z_i .
- (2) Consequently: No Z-jump j_i makes Achilles to reach point x .
- (3) Consequently: Achilles comes to point x from no Z-point.

But the only actions Achilles performs from t_0 is the ω -ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ on the ω -ordered sequence of Z-points $\langle z_i \rangle_{i \in \mathbb{N}}$. So, Achilles can only come from a Z-point as a consequence of a Z-jump. How is then possible Achilles reaches point x at t_b if none of the performed Z-jumps places him there? At this point of the discussion, most infinitists claim that although Achilles comes to point x from no Z-point as a consequence of no Z-jump, it reaches that point at t_b as a consequence of having *completed* the (uncompletable) ω -ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$. As if the completion of the ω -ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ were a place one may come from. But if the completion of the ω -ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ means the completion of the ω -ordered sequence of Z-jumps $\langle s_i \rangle_{i \in \mathbb{N}}$, i.e. that each one of the countably many Z-jumps j_1, j_2, j_3, \dots , *and only them*, have been performed, then it is quite clear that Achilles cannot reach point x at t_b . Simply because no Z-jump j_1, j_2, j_3, \dots ends on the point x . And if no Z-jump j_1, j_2, j_3, \dots , end on the point x and Achilles only performs Z-jumps, then he cannot end on point x either. On the other hand, if the completion of the ω -ordered

³⁰This is the way infinitists pretend to explain how an ω -ordered sequence of actions can be completed: by pairing off two endless sequences, the one of actions the other of instants at which the successive actions are carried out. Thus, in order to end an endless sequence of actions we only need to pair the endless sequence of actions with the endless sequence of instants as which they are performed, as if by pairing off two impossibilities a possibility could result..

sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ were an additional jump then we would have an $(\omega + 1)$ -ordered sequence of jumps rather than an ω -ordered one. But we have proved the ω -ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$ suffices to place Achilles on point x at t_b . It is therefore that ω -ordered sequence $\langle j_i \rangle_{i \in \mathbb{N}}$ which places and does not place Achilles on point x .

Achilles ends his ω -ordered sequence of Z-jumps on point x and this final position is unexplainable because no final jump places him there. And no final jump places him there because no final Z-jumps exists in the ω -ordered sequence of Z-jumps $\langle j_i \rangle_{i \in \mathbb{N}}$. The asymmetry is quite clear: there exists a last effect (to reach point x) but not a last jump causing it. Infinitist, therefore, have to make use of a mysterious last jump by converting the completion of an ω -ordered sequence of jumps in a subsequent additional jump which is different from all previously performed ones. But an ω -ordered sequence of jumps plus an additional last jump is not an ω -ordered sequence of jumps but an $(\omega + 1)$ -ordered one. Thus, this assumed additional jump does not solve the question, because Achilles reaches and does not reaches point x as a consequence of an ω -ordered (not of an $(\omega + 1)$ -ordered) sequence of jumps.

As in the case of Dichotomy II, assume that each Z-point z_i is provided with a mass Z-sensor, being the system of Z-sensors regulated in such a way that, once a Z-sensor is activated, it will be emitting its corresponding laser beam until other Z-sensor be activated and emits its own laser beam. In consequence, once the system is activated there will always be a Z-beam being emitted: the one corresponding to the last activated Z-sensor. So, once activated, it is impossible to turn off the emission of Z-beams. Assume now Achilles performs an ω -ordered sequence of Z-jumps on the ω -ordered sequence of Z-sensitized Z-points. For the same reasons above, Achilles completes this ω -ordered sequence of Z-jumps at t_b . And now the question is: will any laser beam being emitted at t_b ? According to the functioning of the Z-sensors system, once Achilles activates the first Z-sensor by Z-jumping on the first Z-point, it is impossible to turn off the emission of Z-beams. So, at t_b a Z-beam has to be being emitted. Although, on the other hand, no Z-beam can be being emitted at t_b because, if Achilles has completed his uncompletable ω -ordered sequence of Z-jumps, that Z-beam would have to be being emitted by the impossible last Z-sensor of the ω -ordered sequence of Z-sensors. Thus, if Achilles has completed the

uncompletable ω -ordered sequence of Z-jumps, a laser beam will and will not be being emitted by the system of Z-sensors. This seems rather contradictory.

REFERENCES

- [1] Joseph S. Alper and Mark Bridger, *Mathematics, Models and Zeno's Paradoxes*, Synthese **110** (1997), 143 – 166. 5, 10
- [2] ———, *On the Dynamics of Perez Laraudogotia's Supertask*, Synthese **119** (1999), 325 – 337. 7
- [3] Joseph S. Alper, Mark Bridger, John Earman, and John D. Norton, *What is a Newtonian System? The Failure of Energy Conservation and Determinism in Supertasks*, Synthese **124** (2000), 281 – 293. 7
- [4] Aristóteles, *Física*, Gredos, Madrid, 1998. 11
- [5] John Baez, *The Quantum of Area?*, Nature **421** (2003), 702 – 703. 12
- [6] Jacob D. Bekenstein, *La información en un universo holográfico*, Investigación y Ciencia (2003), no. 325, 36 – 43. 12
- [7] Gordon Belot and John Earman, *Pre-socratic quantum gravity*, Physics meets philosophy at the Planck scales (Craig Callender and Nick Huggett, eds.), Cambridge University Press, Cambridge, UK, 2001. 13
- [8] Paul Benacerraf, *Tasks, Super-tasks, and Modern Eleatics*, Journal of Philosophy **LIX** (1962), 765–784. 3
- [9] Henri Bergson, *The Cinematographic View of Becoming*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 59 – 66. 2
- [10] ———, *La evolución creadora*, Espasa Calpe, Madrid, 2004. 2
- [11] JoséA. Bernadete, *Infinity: An essay in Metaphysics*, Oxford University Press, Oxford, 1964. 29
- [12] M. Black, *Achilles and the Tortoise*, Analysis **XI** (1950 - 51), 91 – 101. 3
- [13] Bernard Bolzano, *Les paradoxes de l'infini*, Ed. du Seuil, Paris, 1993. 21
- [14] Florian Cajori, *The History of Zeno's Arguments on Motion*, American Mathematical Monthly **XXII** (1915), 1–6, 38–47, 77–82, 109–115, 143–149, 179,–186, 215–220, 253–258, 292–297, <http://www.matedu.cinvestav.mx/librosydocelec/Cajori.pdf>. 1
- [15] ———, *The Purpose of Zeno's Arguments on Motion*, Isis **III** (1920-21), 7–20. 1
- [16] Georg Cantor, *Über Eine elementare frage der mannigfaltigkeitslehre*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 1, 1891. 23
- [17] ———, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematische Annalen **XLVI** (1895), 481 – 512. 23
- [18] ———, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematische Annalen **XLIX** (1897), 207 – 246. 23
- [19] ———, *Gesammelte Abhandlungen*, Verlag von Julius Springer, Berlin, 1932. 2
- [20] ———, *Contributions to the founding of the theory of transfinite numbers*, Dover, New York, 1955. 2, 28

- [21] ———, *Foundations of a General Theory of Manifolds*, The Theoretical Journal of the National Caucus of Labor Committees **9** (1976), no. 1-2, 69 – 96. [23](#)
- [22] ———, *On The Theory of the Transfinite. Correspondence of Georg Cantor and J. B: Cardinal Franzelin*, Fidelio **III** (1994), no. 3, 97 – 110. [23](#)
- [23] P. Clark and S. Read, *Hypertasks*, Synthese **61** (1984), 387 – 390. [4](#)
- [24] Jonas Cohn, *Histoire de l'infini. Le problème de l'infini dans la pensée occidentale jusqu'à Kant*, LesÉditions du CERF, Paris, 1994. [11](#)
- [25] Giorgio Colli, *Zenón de Elea*, Sexto Piso, Madrid, 2006. [1](#)
- [26] Josep W. Dauben, *Georg Cantor. His mathematics and Philosophy of the Infinite*, Princeton University Press, Princeton, N. J., 1990. [22](#)
- [27] Richard Dedekind, *Quéson y para quésirven los números(was sind Und was sollen die Zahlen(1888))*, Alianza, Madrid, 1998. [21](#)
- [28] John Earman, *Determinism: What We Have Learned and What We Still Don't Know*, Freedom and Determinism (Michael O'Rourke and David Shier, eds.), MIT Press, Cambridge, 2004, pp. 21–46. [7](#)
- [29] John Earman and John D. Norton, *Forever is a Day: Supertasks in Pitowsky and Malament-hogarth Spacetimes*, Philosophy of Science **60** (1993), 22–42. [7](#)
- [30] ———, *Infinite Pains: The Trouble with Supertasks*, Paul Benacerraf: The Philosopher and His Critics (S. Stich, ed.), Blackwell, New York, 1996. [7](#)
- [31] ———, *Comments on Laraudogoitia's 'classical Particle Dynamics, Indeterminism and a Supertask'*, The British Journal for the Philosophy of Science **49** (1998), no. 1, 122 – 133. [7](#)
- [32] JoséL. Fernández Barbón, *Geometría no conmutativa y espaciotiempo cuántico.*, Investigación y Ciencia (2005), no. 342, 60–69. [12](#)
- [33] Richard Feynman, *Superstrings: A Theory of Everything?*, Superstrings: A Theory of Everything? (Paul Davis and Julian Brown, eds.), Cambridge University Press, Cambridge, 1988. [24](#)
- [34] Robert Goldblatt, *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*, Springer-Verlag, New York, 1998. [6](#)
- [35] Brian Green, *El universo elegante*, Editorial Crítica, Barcelona, 2001. [12](#), [24](#)
- [36] ———, *The Fabric of the Cosmos. Space. Time. And the Texture of Reality*, Alfred A. Knopf, New York, 2004. [12](#), [24](#)
- [37] Adolf Grünbaum, *Modern Science and Refutation of the Paradoxes of Zeno*, The Scientific Monthly **LXXXI** (1955), 234–239. [8](#)
- [38] ———, *Modern Science and Zeno's Paradoxes*, George Allen And Unwin Ltd, London, 1967. [8](#), [11](#), [17](#)
- [39] ———, *Modern Science and Refutation of the Paradoxes of Zeno*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 164 – 175. [7](#), [8](#)
- [40] ———, *Modern Science and Zeno's Paradoxes of Motion*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 200 – 250. [7](#), [8](#)

- [41] ———, *Zeno's Metrical Paradox of Extension*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 176 – 199. [7](#), [8](#)
- [42] Thomas Heath, *A History of Greek Mathematics*, Dover Publications Inc, New York, 1981. [11](#)
- [43] Georg Wilhelm Friedrich Hegel, *Lectures on the History of Philosophy. Greek Philosophy to Plato*, vol. 1, University of Nebraska Press, Lincoln, 1995. [2](#)
- [44] James M. Henle and Eugene M. Kleinberg, *Infinitesimal Calculus*, Dover Publications Inc., Mineola, New York, 2003. [6](#)
- [45] M. L. Hogarth, *Does General Relativity Allow an Observer to view an Eternity in a Finite Time?*, Foundations of Physics Letters **5** (1992), 173 – 181. [7](#)
- [46] Gerard't Hooft, *Partículas elementales*, Crítica, Barcelona, 1991. [24](#)
- [47] Nick Huggett, *Zeno's Paradoxes*, The Stanford Encyclopaedia of Philosophy (Summer 2004 Edition) (Edward N. Zalta(ed.), ed.), Stanford University, <http://plato.stanford.edu/archives/sum2004/entries/paradox-zeno/>, 2004. [1](#), [10](#)
- [48] William James, *Some Problems of Philosophy*, Bison Books, Univ. of Nebraska Press, Lincoln, NE, 1996. [2](#)
- [49] José Ferreirós, *Matemáticas y platonismo(s)*, Gaceta de la Real Sociedad Matemática Española **2** (1999), 446–473. [16](#)
- [50] H. Jerome Keisler, *Elementary Calculus. An Infinitesimal Approach*, second ed., Author, <http://www.wisc.edu/keisler/keislercalc.pdf>, September 2002. [6](#)
- [51] Seth Loyd and Y. Jack Ng, *Computación en agujeros negros*, Investigación y Ciencia (Scientific American) (2005), no. 340, 59 – 67. [12](#)
- [52] Peter Lynds, *Time and Classical and Quantum Mechanics: Indeterminacy vs. Discontinuity*, Foundations of Physics Letters **16** (2003), 343 – 355. [9](#)
- [53] Peter Lynds, *Zeno's Paradoxes: A Timely Solution*, philsci-archives (3003), 1 – 9, <http://philsci-archives.pitt.edu/archive/00001197>. [9](#)
- [54] William I. Macloughlin, *Thomson's Lamp is Dysfunctional*, Synthese **116** (1998), no. 3, 281 – 301. [5](#)
- [55] Mathieu Marion, *Wittgenstein, finitism and the foundations of mathematics*, Clarendon Press Oxford, Oxford, 1998. [18](#)
- [56] Joseph Mazur, *The Motion Paradox*, Dutton, 2007. [1](#)
- [57] William I. McLaughlin, *Una resolución de las paradojas de Zenón*, Investigación y Ciencia (Scientific American) (1995), no. 220, 62 – 68. [5](#), [8](#)
- [58] William I. McLaughlin and Silvia L. Miller, *An Epistemological Use of non-standard Analysis to Answer Zeno's Objections Against Motion*, Synthese **92** (1992), no. 3, 371 – 384. [5](#), [8](#)
- [59] J. E. McTaggart, *The unreality of time*, Mind **17** (1908), 457 – 474. [14](#)
- [60] Chris Mortensen, *Change*, Stanford Encyclopaedia of Philosophy (E. N. Zalta, ed.), Stanford University, URL = <http://plato.stanford.edu>, 2002. [14](#)
- [61] John D. Norton, *A Quantum Mechanical Supertask*, Foundations of Physics **29** (1999), 1265 – 1302. [7](#)

- [62] Alba Papa-Grimaldi, *Why mathematical solutions of Zeno's paradoxes miss the point: Zeno's one and many relation and Parmenides prohibition*, *The Review of Metaphysics* **50** (1996), 299–314. [10](#), [15](#)
- [63] Jon Pérez Laraudogoitia, *A Beautiful Supertask*, *Mind* **105** (1996), 49–54. [7](#)
- [64] ———, *Classical Particle Dynamics, Indeterminism and a Supertask*, *British Journal for the Philosophy of Science* **48** (1997), 49 – 54. [7](#)
- [65] ———, *Infinity Machines and Creation Ex Nihilo*, *Synthese* **115** (1998), 259 – 265. [7](#)
- [66] ———, *Why Dynamical Self-excitation is Possible*, *Synthese* **119** (1999), 313 – 323. [7](#)
- [67] ———, *Supertasks*, *The Stanford Encyclopaedia of Philosophy* (E. N. Zalta, ed.), Stanford University, URL = <http://plato.stanford.edu>, 2001. [7](#), [10](#)
- [68] Jon Pérez Laraudogoitia, Mark Bridger, and Joseph S. Alper, *Two Ways of Looking at a Newtonian Supertask*, *Synthese* **131** (2002), no. 2, 157 – 171. [7](#)
- [69] I. Pitowsky, *The Physical Church Thesis and Physical Computational Complexity*, *Iyyun* **39** (1990), 81 – 99. [7](#)
- [70] Henri Poincaré, *Dernières pensées*, Flammarion, Paris, 1913. [19](#)
- [71] Rudy Rucker, *Infinity and the Mind*, Princeton University Press, Princeton, 1995. [2](#)
- [72] Bertrand Russell, *The Problem of Infinity Considered Historically*, *Zeno's Paradoxes* (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 45 – 58. [2](#)
- [73] Wesley C. Salmon, *Introduction*, *Zeno's Paradoxes* (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis, Cambridge, 2001, pp. 5 – 44. [1](#), [7](#), [10](#), [11](#)
- [74] Z. K. Silagadze, *Zeno meets modern science*, *Philsci-archieve* (2005), 1–40. [7](#), [10](#)
- [75] Lee Smolin, *Three roads to quantum gravity. Anew understanding of space, time and the universe*, Phoenix, London, 2003. [12](#)
- [76] Lee Smolin, *Átomos del espacio y del tiempo*, *Investigación y Ciencia*(Scientific American) (2004), no. 330, 58 – 67. [12](#)
- [77] K. D. Stroyan, *Foundations of Infinitesimal Calculus*, Academic Press, Inc, New York, 1997. [6](#)
- [78] Leonard Susskind, *Los agujeros negros y la paradoja de la información*, *Investigación y Ciencia*(Scientific American) (1997), no. 249, 12 – 18. [12](#)
- [79] James F. Thomson, *Tasks and Supertasks*, *Analysis* **15** (1954), 1–13. [3](#)
- [80] ———, *Comments on Professor Benacerraf's Paper*, *Zeno's Paradoxes* (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 130 – 138. [3](#)
- [81] Gabriele Veneziano, *El universo antes de la Gran Explosión*, *Investigación y Ciencia*(Scientific American) (2004), no. 334, 58 – 67. [12](#)
- [82] Gregory Vlastos, *Zeno's Race Course*, *Journal of the History of Philosophy* **IV** (1966), 95–108. [25](#)
- [83] ———, *Zeno of Elea*, *The Encyclopaedia of Philosophy* (Paul Edwards, ed.), McMillan and Free Press, New York, 1967. [1](#)
- [84] David Foster Wallace, *Everything and more. Acompact history of infinity*, Orion Books Ltd., London, 2005. [1](#)

-
- [85] H. Weyl, *Philosophy of Mathematics and Natural Sciences*, Princeton University Press, Princeton, 1949. 7
 - [86] Alfred North Whitehead, *Essays in Science and Philosophy*, Philosophical Library, New York, 1948. 2
 - [87] Alfred North Whitehead, *Process and Reality*, The Free Press, New York, 1978. 2
 - [88] J. O. Wisdom, *Achilles on a Physical Racecourse*, *Analysis* **XII** (1951-52), 67–72. 3
 - [89] Ludwig Wittgenstein, *Remarks on the Foundations of Mathematics*, Basil Blackwell, Oxford, 1978. 19
 - [90] F. J. Ynduráin, *Electrones, neutrinos y quarks*, Crítica, Barcelona, 2001. 24
 - [91] Mark Zangari, *Zeno, Zero and Indeterminate Forms: Instants in the Logic of Motion*, *Australasian Journal of Philosophy* **72** (1994), 187–204. 8